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A Note on Spectral Curve for the Periodic Homogeneous XYZ-Spin Chain [1]

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Abstract

We discuss the construction of the spectral curve and the action integrals for the "elliptic" XYZ spin chain of the length N_c. Our analysis can reflect the integrable structure behind the "elliptic" $\mathcal{N} = 2$ supersymmetric QCD with $N_f = 2N_c$.

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In recent paper [1] we argued that the integrable counterpart [2]-[7] of the low-energy $4d$ $\mathcal{N}=2$ SUSY Yang-Mills theory with N_f fundamental matter hypermultiplets [8]-[10] can be searched among integrable spin chains. A reasonable suggestion was made in [1] about the "rational" case of $N_f < 2N_c$ (see also [5, 6]). However, the story can not be complete without studying the most intriguing "elliptic" case of $N_f = 2N_c$, when the $4d$ theory is UV-finite and possesses an extra *dimensionless* parameter: the UV non-abelian coupling constant $\tau = \frac{8\pi i}{e^2} + \frac{\theta}{\pi}$. Before addressing to the physical problems, some preliminary technical work should be done, and we start such work in the present paper by analyzing relevant aspects of the theory of XYZ spin chains. The $4d$ interpretation of this theory will not be directly addressed here and postponed to a separate paper.

The main ingredients of interest for us are the spectral curve $\mathcal{C}$, associated with the spin chain model, and the "action integrals" $a_i = \oint_{A_i} dS$, $a_D^i = \oint_{B^i} dS$, or the moduli-dependence of the cohomology class of the "generating" 1-form dS defined by the property

$$\frac{\partial dS}{\partial\{moduli\}} \cong \text{ holomorphic 1-form} \tag{1}$$

The main information on the XYZ models can be found in refs. [11]-[13] (see also [14] for its continuum limit).

1 Toda chain: $N_c \times N_c$ versus 2×2 representation

We begin our analysis from the simplest Toda-chain model, which in the framework of the Seiberg-Witten solutions corresponds to the $4d$ *pure* gauge $\mathcal{N}=2$ supersymmetric Yang-Mills theory. The periodic problem in this model can be formulated in two different ways, which will be further deformed into two different directions. These deformations are hypothetically related to the two different deformations of the $4d$ theory by adding the adjoint and fundamental matter $\mathcal{N}=2$ hypermultiplets correspondingly.

The Toda chain system can be defined by the equations of motion

$$\frac{\partial q_i}{\partial t} = p_i \qquad \frac{\partial p_i}{\partial t} = e^{q_{i+1}-q_i} - e^{q_i - q_{i-1}} \tag{2}$$

where one assumes (for the periodic problem with the "period" N_c) that $q_{i+N_c} = q_i$ and $p_{i+N_c} = p_i$. It is an integrable system, with N_c Poisson-commuting Hamiltonians, $h_1^{TC} = \sum p_i$, $h_2^{TC} = \sum \left(\frac{1}{2}p_i^2 + e^{q_i - q_{i-1}}\right)$, ... The generation function for these Hamiltonians can be written in terms of a Lax operator and as was already mentioned, the Toda chain possesses two essentially different formulations of this kind.

In the first version (which can be considered as a limiting case of Hitchin system [15]), the Lax operator is the $N_c \times N_c$ matrix-valued 1-form,

$$\mathcal{L}^{TC}(w)\frac{dw}{w} = \begin{pmatrix} p_1 & e^{\frac{1}{2}(q_1-q_2)} & 0 & & we^{\frac{1}{2}(q_1-q_{N_c})} \\ e^{\frac{1}{2}(q_2-q_1)} & p_2 & e^{\frac{1}{2}(q_2-q_3)} & \dots & 0 \\ 0 & e^{\frac{1}{2}(q_3-q_2)} & p_3 & & 0 \\ & & \dots & & \\ \frac{1}{w}e^{\frac{1}{2}(q_{N_c}-q_1)} & 0 & 0 & & p_{N_c} \end{pmatrix} \frac{dw}{w} \tag{3}$$

defined on the two-punctured sphere. The Poisson brackets $\{p_i, q_j\} = \delta_{ij}$ imply that

$$\left\{\mathcal{L}^{TC}(w) \overset{\otimes}{,} \mathcal{L}^{TC}(w')\right\} = \left[\mathcal{R}(w,w'), \mathcal{L}^{TC}(w) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}^{TC}(w')\right] \tag{4}$$

with the *numeric* trigonometric $\mathcal{R}$-matrix [17], and the eigenvalues of the Lax operator defined from the spectral equation

$$\det_{N_c \times N_c}\left(\mathcal{L}^{TC}(w) - \lambda\right) = 0 \tag{5}$$

are Poisson-commuting with each other. Moreover, the set of the action integrals along the cycles $\hat{\gamma}_a$ in the $2N_c$-dimensional phase space of the system can be reproduced by the integrals of the generating "eigenvalue" 1-form $dS^{TC} \cong \lambda \frac{dw}{w}$ along some non-contractable cycles γ_a on the spectral curve $\mathcal{C}^{TC}$ defined by eq.(5):

$$I_{\gamma_a} = \oint_{\hat{\gamma}_a} \sum_{i=1}^{N_c} p_i dq_i = \oint_{\gamma_a} dS \tag{6}$$

The generating differential dS plays an essential role in the theory of the (finite - gap) integrable systems. It gives rise to the symplectic form on the phase space of the finite-gap solutions, describes their Whitham-like deformations [18, 2, 5] and allows one to find explicitely the action-angle variables in the framework of so-called "separation of variables" [11] and Hitchin formalism [15] (see also [16]). Substituting the explicit expression (3) into (5), one gets [19]:

$$w + \frac{1}{w} = 2P_{N_c}(\lambda) \tag{7}$$

where $P_{N_c}(\lambda)$ is a polynomial of degree N_c, with the coefficients being the Schur polynomials of the Hamiltonians $h_k = \sum_{i=1}^{N_c} p_i^k + \ldots$:

$$\begin{aligned} P_{N_c}(\lambda) &= \sum_{k=0}^{N_c} \mathcal{S}_{N_c-k}(h)\lambda^{N_c} = \\ &= \left(\lambda^{N_c} + h_1\lambda^{N_c-1} + \frac{1}{2}(h_2 - h_1^2)\lambda^{N_c-2} + \ldots\right) \end{aligned} \tag{8}$$

The spectral equation depends only on the mutually Poisson-commuting combinations of the dynamical variables – the Hamiltonians – parametrizing (a subspace in the) moduli space of the complex structures of the hyperelliptic curves $\mathcal{C}^{TC}$ of genus $N_c - 1$.

An alternative description of the same system involves (a chain of) 2×2 matrices [12],

$$L_i^{TC}(\lambda) = \begin{pmatrix} p_i + \lambda & e^{q_i} \\ e^{-q_i} & 0 \end{pmatrix}, \qquad i = 1, \ldots, N_c \tag{9}$$

obeying the *quadratic* r-matrix Poisson relations [11]

$$\left\{L_i^{TC}(\lambda) \overset{\otimes}{,} L_j^{TC}(\lambda')\right\} = \delta_{ij}\left[r(\lambda - \lambda'), L_i^{TC}(\lambda) \otimes L_j^{TC}(\lambda')\right] \tag{10}$$

with the (i-independent!) numerical rational r-matrix satisfying the classical Yang-Baxter equation $r(\lambda) = \frac{1}{\lambda}\sum_{a=1}^{3} \sigma_a \otimes \sigma^a$. As a consequence, the transfer matrix (generally defined for the inhomogeneous lattice with inhomogenities λ_i's)

$$T_{N_c}(\lambda) = \prod_{1 \geq i \geq N_c}^{\curvearrowleft} L_i(\lambda - \lambda_i) \tag{11}$$

satisfies the same Poisson relation

$$\left\{T_{N_c}(\lambda) \overset{\otimes}{,} T_{N_c}(\lambda')\right\} = [r(\lambda - \lambda'), T_{N_c}(\lambda) \otimes T_{N_c}(\lambda')] \tag{12}$$

and the integrals of motion of the Toda chain are generated by another form of spectral equation

$$\det_{2\times 2}\left(T^{TC}_{N_c}(\lambda) - w\right) = w^2 - w\mathrm{Tr}T^{TC}_{N_c}(\lambda) + 1 = 0 \tag{13}$$

or

$$w + \frac{1}{w} = \mathrm{Tr}T^{TC}_{N_c}(\lambda) \tag{14}$$

(We used the fact that $\det_{2\times 2} L^{TC}(\lambda) = 1$ leads to $\det_{2\times 2} T^{TC}_{N_c}(\lambda) = 1$.) The r.h.s. of (14) is a polynomial of degree N_c in λ, with the coefficients being the integrals of motion since

$$\begin{gathered}\left\{\mathrm{Tr}T_{N_c}(\lambda), \mathrm{Tr}T_{N_c}(\lambda')\right\} = \mathrm{Tr}\left\{T_{N_c}(\lambda) \overset{\otimes}{,} T_{N_c}(\lambda')\right\} = \\ = \mathrm{Tr}\left[r(\lambda-\lambda'), T_{N_c}(\lambda)\otimes T_{N_c}(\lambda')\right] = 0\end{gathered} \tag{15}$$

For the particular choice of L-matrix (9), the inhomogenities of the chain, λ_i, can be absorbed into the redefinition of the momenta $p_i \to p_i - \lambda_i$. It is possible to establish a straightforward relation between two representations (5) and (13) (see [1] for details).

In what follows we consider possible elliptic deformations of two Lax representations of the Toda chain. The deformation of the $N_c \times N_c$ representation provides the Calogero-Moser model (discussed in the context of the Seiberg-Witten approach in [4]), while the deformation of the spin-chain 2×2 representation gives rise to the Sklyanin XYZ model.

2 Elliptic deformation of the $N_c \times N_c$ representation: the Calogero-Moser model

The $N_c \times N_c$ matrix-valued Lax 1-form for the $GL(N_c)$ Calogero system is [20]

$$\begin{gathered}\mathcal{L}^{Cal}(\xi)d\xi = \left(\mathbf{pH} + \sum_{\boldsymbol{\alpha}} F(\mathfrak{q}\boldsymbol{\alpha}|\xi)E_{\boldsymbol{\alpha}}\right)d\xi = \\ = \begin{pmatrix} p_1 & F(\mathfrak{q}_1-\mathfrak{q}_2|\xi) & \dots & F(\mathfrak{q}_1-\mathfrak{q}_{N_c}|\xi) \\ F(\mathfrak{q}_2-\mathfrak{q}_1|\xi) & p_2 & \dots & F(\mathfrak{q}_2-\mathfrak{q}_{N_c}|\xi) \\ & & \dots & \\ F(\mathfrak{q}_{N_c}-\mathfrak{q}_1|\xi) & F(\mathfrak{q}_{N_c}-\mathfrak{q}_2|\xi) & \dots & p_{N_c}\end{pmatrix} d\xi\end{gathered} \tag{16}$$

The function $F(\mathfrak{q}|\xi) = \frac{g}{\omega}\frac{\sigma(\mathfrak{q}+\xi)}{\sigma(\mathfrak{q})\sigma(\xi)}e^{\zeta(\mathfrak{q})\xi}$ and, thus, the Lax operator $\mathcal{L}(\xi)d\xi$ is defined on the elliptic curve $E(\tau)$ (complex torus with periods ω, ω' and modulus $\tau = \frac{\omega'}{\omega}$). The Calogero coupling constant is $\frac{g^2}{\omega^2} \sim m^2$, where in the $4d$ interpretation m plays the role of the mass of the adjoint matter $\mathcal{N} = 2$ hypermultiplet breaking $\mathcal{N} = 4$ SUSY down to $\mathcal{N} = 2$.

The spectral curve $\mathcal{C}^{Cal}$ for the $GL(N_c)$ Calogero system is given by:

$$\det_{N_c\times N_c}\left(\mathcal{L}^{Cal}(\xi) - \lambda\right) = 0 \tag{17}$$

The periods a_i and a_i^D are again the integrals of the generating 1-differential

$$dS^{Cal} \cong \lambda d\xi \tag{18}$$

along the non-contractable contours on $\mathcal{C}^{Cal}$. Integrability of the Calogero system is implied by the Poisson structure of the form

$$\left\{\mathcal{L}(\xi) \overset{\otimes}{,} \mathcal{L}(\xi')\right\} = \left[\mathcal{R}_{12}^{Cal}(\xi,\xi'),\ \mathcal{L}(\xi)\otimes\mathbf{1}\right] - \left[\mathcal{R}_{21}^{Cal}(\xi,\xi'),\ \mathbf{1}\otimes\mathcal{L}(\xi')\right] \tag{19}$$

with the *dynamical* elliptic $\mathcal{R}$-matrix [21], guaranteeing that the eigenvalues of the matrix $\mathcal{L}$ are in involution.

In order to recover the Toda-chain system, one takes the double-scaling limit [22], when $g \sim m$ and $-i\tau$ both go to infinity (and $\mathfrak{q}_i - \mathfrak{q}_j = \frac{1}{2}\left[(i-j)\log g + (q_i - q_j)\right]$) so that the dimensionless coupling τ gets substituted by a dimensional parameter $\Lambda^{N_c} \sim m^{N_c} e^{i\pi\tau}$. In this limit, the elliptic curve $E(\tau)$ degenerates into the (two-punctured) Riemann sphere with coordinate $w = e^{\xi}e^{i\pi\tau}$ so that

$$dS^{Cal} \to dS^{TC} \cong \lambda \frac{dw}{w} \tag{20}$$

The Lax operator of the Calogero system turns into that of the N_c-periodic Toda chain (3):

$$\mathcal{L}^{Cal}(\xi)d\xi \to \mathcal{L}^{TC}(w)\frac{dw}{w} \tag{21}$$

and the spectral curve acquires the form of (5). In contrast to the Toda case, (17) can *not* be rewritten in the form (7) and peculiar w-dependence of the spectral equation (5) is not preserved by embedding of Toda into Calogero-Moser particle system. However, the form (7) can be naturally preserved by the alternative deformation of the Toda-chain system when it is considered as (a particular case of) a spin-chain model.

To deal with the "elliptic" deformations of the Toda chain below, we will use a non-standard normalization of the Weierstrass $\wp$-function defined by

$$\wp(\xi|\tau) = \sum_{m,n=-\infty}^{+\infty} \frac{1}{(\xi+m+n\tau)^2} - \sum_{m,n=-\infty}^{+\infty}{}' \frac{1}{(m+n\tau)^2} \tag{22}$$

so that it is double periodic in ξ with periods 1 and $\tau = \frac{\omega'}{\omega}$ (that differs from a standard definition by a factor of ω^{-2} and by rescaling $\xi \to \omega\xi$). According to (22), the values of $\wp(\xi|\tau)$ in the half-periods, $e_a = e_a(\tau)$, $a = 1,2,3$, are also the functions only of τ – again differing by a factor of ω^{-2} from the conventional definition.

The complex torus $E(\tau)$ can be defined as $\mathbf{C}/\mathbf{Z}\oplus\tau\mathbf{Z}$ with a "flat" co-ordinate ξ defined *modulo*$(1,\tau)$. Alternatively, any torus (with a marked point) can be described as elliptic curve

$$\begin{gathered} y^2 = (x-e_1)(x-e_2)(x-e_3) \\ x = \wp(\xi) \qquad y = \frac{1}{2}\wp'(\xi) \end{gathered} \tag{23}$$

and the canonical holomorphic 1-differential is

$$d\xi = 2\frac{dx}{y} \tag{24}$$

There are three interesting degeneration limits:

– <u>rational limit</u>: both periods $\omega,\omega' \to \infty$, ξ scales as $\xi = \omega^{-1}\zeta$ with $\tau = \frac{\omega'}{\omega}$ and ζ remain finite. Then:

$$x = \wp(\xi) = \frac{\omega^2}{\zeta^2}(1+o(\omega^{-1})) \qquad y = \frac{1}{2}\wp'(\xi) = -\frac{\omega^3}{\zeta^3}(1+o(\omega^{-1})) \tag{25}$$

In two other limits $\tau \to +i\infty$, i.e. $q = e^{i\pi\tau} \to 0$.

– <u>trigonometric limit</u>: ξ remains finite as $q \to 0$

$$x = \wp(\xi) = -\frac{1}{3} + \frac{1}{\sin^2\pi\xi} + o(q) \qquad y = \frac{1}{2}\wp'(\xi) = -\pi\frac{\cos\pi\xi}{\sin^3\pi\xi} + o(q) \tag{26}$$

– <u>double-scaling limit</u>: $\xi = \log(qw)$, the branch points

$$e_{1,2} \to -\frac{1}{3} \pm 8q + o(q^2) \qquad e_3 \to +\frac{2}{3} + o(q^2) \tag{27}$$

and

$$x = \wp(\xi) = -\frac{1}{3} + 4q(w + w^{-1}) + o(q^2) \qquad y = \frac{1}{2}\wp'(\xi) = 4q(w - w^{-1}) + o(q^2) \tag{28}$$

and

$$d\xi = \frac{dw}{w}(1 + \mathcal{O}(q)) \tag{29}$$

In the simplest example of $N_c = 2$, the spectral curve $\mathcal{C}^{Cal}$ has genus 2. Indeed, in this particular case, eq.(17) turns into

$$\lambda^2 = h_2 - \frac{g^2}{\omega^2}\wp(\xi) = h_2 - \frac{g^2}{\omega^2}x \tag{30}$$

This equation says that with any value of x one associates two points of $\mathcal{C}^{Cal}$, $\lambda = \pm\sqrt{h_2 - \frac{g^2}{\omega^2}x}$, i.e. it describes $\mathcal{C}^{cal}$ as a double covering of the elliptic curve $E(\tau)$ ramified at the points $x = \left(\frac{\omega}{g}\right)^2 h_2$ and $x = \infty$. In fact, since x is an elliptic coordinate on $E(\tau)$ (when elliptic curve is treated as a double covering over the Riemann sphere CP^1), $x = \left(\frac{\omega}{g}\right)^2 h_2$ corresponds to a *pair* of points on $E(\tau)$ distinguished by the sign of y. This would be true for $x = \infty$ as well, but $x = \infty$ is one of the branch points in our parametrization (23) of $E(\tau)$. Thus, the *two* cuts between $x = \left(\frac{\omega}{g}\right)^2 h_2$ and $x = \infty$ on every sheet of $E(\tau)$ touching at the common end at $x = \infty$ become a *single* cut between $\left(\left(\frac{\omega}{g}\right)^2 h_2, +\right)$ and $\left(\left(\frac{\omega}{g}\right)^2 h_2, -\right)$. Therefore, we can consider the spectral curve $\mathcal{C}^{Cal}$ as two tori $E(\tau)$ glued along one cut, i.e. $\mathcal{C}^{Cal}_{N_c=2}$ is a curve of genus 2.

Analytically the curve $\mathcal{C}^{Cal}$ for $N_c = 2$ can be described by the pair of equations:

$$\begin{aligned} y^2 &= \prod_{a=1}^{3}(x - e_a), \\ \lambda^2 &= h_2 - \frac{g^2}{\omega^2}x \end{aligned} \tag{31}$$

Occasionally, it turns out to be a hyperelliptic curve (for $N_c = 2$ only!) after substituting in (31) x from the second equation to the first one.

Two holomorphic 1-differentials on $\mathcal{C}^{Cal}$ can be chosen to be

$$v = \frac{dx}{y} \sim \frac{\lambda d\lambda}{y} \qquad V = \frac{dx}{y\lambda} \sim \frac{d\lambda}{y} \tag{32}$$

so that

$$dS \cong \lambda d\xi = \sqrt{h_2 - \frac{g^2}{\omega^2}\wp(\xi)}d\xi = \frac{dx}{y}\sqrt{h_2 - \frac{g^2}{\omega^2}x} \tag{33}$$

It is easy to check the basic property (1):

$$\frac{\partial dS}{\partial h_2} \cong \frac{1}{2}\frac{dx}{y\lambda} \tag{34}$$

The fact that only one of two holomorphic 1-differentials (32) appears at the r.h.s. is related to their different parity with respect to the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ symmetry of $\mathcal{C}^{Cal}$: $y \to -y$ and $\lambda \to -\lambda$. Since dS has certain parity, its integrals along the two of four elementary non-contractable cycles on $\mathcal{C}^{Cal}$ automatically vanish leaving only two non-vanishing quantities a and a_D, as necessary for the $4d$ interpretation [4]. Moreover, these two non-vanishing integrals can be actually evaluated in terms of the "reduced" genus-*one* curve

$$Y^2 = (y\lambda)^2 = \left(h_2 - \frac{g^2}{\omega^2}x\right)\prod_{a=1}^{3}(x - e_a), \tag{35}$$

with $dS \cong \left(h_2 - \frac{g^2}{\omega^2}x\right)\frac{dx}{Y}$. Since now $x = \infty$ is no longer a ramification point, dS obviously has simple poles at $x = \infty$ (at two points on the two sheets of $\mathcal{C}^{Cal}_{reduced}$) with the residues $\pm\frac{g}{\omega} \sim \pm m$.

The opposite limit of the Calogero-Moser system with vanishing coupling constant $g^2 \sim m^2 \to 0$ corresponds to the $\mathcal{N} = 4$ SUSY Yang-Mills theory with identically vanishing β-function. The corresponding integrable system is a collection of *free* particles and the generating differential $dS \cong \sqrt{h_2} \cdot d\xi$ is just a *holomorphic* differential on $E(\tau)$.

3 Spin-chain (magnetic) models and Sklyanin algebra.

This class of integrable models is based on the quadratic Poisson structure (10) for an $n \times n$ matrix $L(\lambda)$, which implies the existence of the monodromy matrix (11) with the Poisson-commuting eigenvalues.

The spectral curve for the *periodic inhomogeneous* spin chain is given by:

$$\det_{n\times n}\left(T_{N_c}(\lambda) - \tilde{w}\right) = 0 \tag{36}$$

and the generating 1-differential is

$$\begin{gathered} dS \cong \lambda\frac{dw}{w} \\ w = \tilde{w}\cdot\det T_{N_c}(\lambda)^{-1/n} \end{gathered} \tag{37}$$

In the particular case of $n = 2$ ($sl(2)$ spin chains), the spectral equation (36) acquires the form:

$$\tilde{w} + \frac{\det_{2\times 2} T_{N_c}(\lambda)}{\tilde{w}} = \mathrm{Tr}T_{N_c}(\lambda) \tag{38}$$

and this peculiar form of w-dependence suggests [5, 1] that the periodic $sl(2)$ spin chains are related to the solution to the $\mathcal{N} = 2$ supersymmetric QCD.

The most general theory of this sort is known as Sklyanin XYZ spin chain with the elementary L-operator defined on the elliptic curve $E(\tau)$ and is explicitly given by (see [12] and references therein):

$$L^{Skl}(\xi) = S^0\mathbf{1} + i\frac{g}{\omega}\sum_{a=1}^{3} W_a(\xi)S^a\sigma_a \tag{39}$$

where

$$\begin{gathered} W_a(\xi) = \sqrt{e_a - \wp(\xi|\tau)} = i\frac{\theta'_{11}(0)\theta_{a+1}(\xi)}{\theta_{a+1}(0)\theta_{11}(\xi)} \\ \theta_2 \equiv \theta_{01}, \quad \theta_3 \equiv \theta_{00}, \quad \theta_4 \equiv \theta_{10} \end{gathered} \tag{40}$$

Let us note that our spectral parameter ξ is connected with the standard one u [12] by the relation $u = 2K\xi$, where $K \equiv \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2\sin^2 t}} = \frac{\pi}{2}\theta^2_{00}(0)$, $k^2 \equiv \frac{e_1-e_2}{e_1-e_3}$ so that $K \to \frac{\pi}{2}$ as $q \to 0$. This factor results into additional multiplier π in the trigonometric functions in the limiting cases below.

The Lax operator (39) satisfies the Poisson relation (10) with the numerical *elliptic* r-matrix $r(\xi) = i\frac{g}{\omega}\sum_{a=1}^{3} W_a(\xi)\sigma_a\otimes\sigma_a$, which implies that S^0, S^a form the (classical) Sklyanin algebra [23, 11]:

$$\begin{gathered} \left\{S^a, S^0\right\} = 2i\left(\frac{g}{\omega}\right)^2(e_b - e_c)\,S^bS^c \\ \left\{S^a, S^b\right\} = 2iS^0S^c \end{gathered} \tag{41}$$

with the obvious notation: abc is the triple 123 or its cyclic permutations.

The coupling constant $\frac{g}{\omega}$ can be eliminated by simultaneous rescaling of the S-variables and the symplectic form:

$$S^a = \frac{\omega}{g}\hat{S}^a \qquad S^0 = \hat{S}^0 \qquad \{\ ,\ \} \to \frac{g}{\omega}\{\ ,\ \} \tag{42}$$

Then

$$L(\xi) = \hat{S}^0 \mathbf{1} + i\sum_{a=1}^{3} W_a(\xi)\hat{S}^a\sigma_a \tag{43}$$

$$\begin{gathered}\left\{\hat{S}^a,\ \hat{S}^0\right\} = 2i\left(e_b - e_c\right)\hat{S}^b\hat{S}^c\\ \left\{\hat{S}^a,\ \hat{S}^b\right\} = 2i\hat{S}^0\hat{S}^c\end{gathered} \tag{44}$$

In parallel with (25)-(28), one can distinguish three interesting limits of the Sklyanin algebra.
— rational limit. Both $\omega, \omega' \to \infty$, then (41) turns into

$$\begin{gathered}\{S^a, S^0\} = 0\\ \{S^a, S^b\} = 2i\epsilon^{abc}S^0S^c\end{gathered} \tag{45}$$

i.e. S^0 itself becomes a Casimir operator (constant), while the remaining S^a form a classical angular-momentum (spin) vector. The corresponding

$$L_{XXX}(\zeta) = \mathbf{1} - \frac{g}{\zeta}\mathbf{S}\cdot\boldsymbol{\sigma} \tag{46}$$

describes the XXX spin chain considered in detail in [1] with the rational r-matrix.
— trigonometric limit. As $\tau \to +i\infty$ or $q \to 0$, the Sklyanin algebra (44) transforms to

$$\begin{gathered}\{\hat{S}^3, \hat{S}^0\} = 32iq\hat{S}^1\hat{S}^2 + \mathcal{O}(q) \to 0\\ \{\hat{S}^1, \hat{S}^0\} = -2i\hat{S}^2\hat{S}^3 + \mathcal{O}(q)\\ \{\hat{S}^2, \hat{S}^0\} = 2i\hat{S}^3\hat{S}^1 + \mathcal{O}(q)\\ \{\hat{S}^1, \hat{S}^2\} = 2i\hat{S}^0\hat{S}^3 + \mathcal{O}(q)\\ \{\hat{S}^1, \hat{S}^3\} = -2i\hat{S}^0\hat{S}^2 + \mathcal{O}(q)\\ \{\hat{S}^2, \hat{S}^3\} = 2i\hat{S}^0\hat{S}^1 + \mathcal{O}(q)\end{gathered} \tag{47}$$

The corresponding Lax matrix is

$$L_{XXZ} = \hat{S}^0\mathbf{1} - \frac{1}{\sin\pi\xi}\left(\hat{S}^1\sigma_1 + \hat{S}^2\sigma_2 + \cos\pi\xi\hat{S}^3\sigma_3\right) \tag{48}$$

and r-matrix

$$r(\xi) = \frac{i}{\sin\pi\xi}\left(\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2 + \cos\pi\xi\sigma_3\otimes\sigma_3\right) \tag{49}$$

— double-scaling limit. Using (27) and (28), we find that

$$\begin{gathered}\sqrt{e_{1,2} - \wp(\xi)} = 2\sqrt{q}\sqrt{w + \frac{1}{w} \pm 2} + \mathcal{O}(q) = 2\sqrt{q}\left(\sqrt{w} \pm \frac{1}{\sqrt{w}}\right) + \mathcal{O}(q)\\ \sqrt{e_3 - \wp(\xi)} = 1 + \mathcal{O}(q)\end{gathered} \tag{50}$$

and, therefore, the Sklyanin algebra (44), after the rescaling $\hat{S}^{1,2} = \frac{1}{4\sqrt{q}}\bar{S}^{1,2}$, acquires the form

$$
\begin{aligned}
\{\bar{S}^3, \bar{S}^0\} &= 2i\bar{S}^1\bar{S}^2 \\
\{\bar{S}^1, \bar{S}^0\} &= -2i\bar{S}^2\bar{S}^3 + \mathcal{O}(q) \\
\{\bar{S}^2, \bar{S}^0\} &= 2i\bar{S}^3\bar{S}^1 + \mathcal{O}(q) \\
\{\bar{S}^1, \bar{S}^2\} &= 32iq\bar{S}^0\bar{S}^3 + \mathcal{O}(q) \to 0 \\
\{\bar{S}^1, \bar{S}^3\} &= -2i\bar{S}^0\bar{S}^2 + \mathcal{O}(q) \\
\{\bar{S}^2, \bar{S}^3\} &= 2i\bar{S}^0\bar{S}^1 + \mathcal{O}(q)
\end{aligned}
\tag{51}
$$

with the Lax matrix

$$
L_{ds} = \bar{S}^0\mathbf{1} + i\bar{S}^3\sigma_3 + \frac{i}{2}\left(\sqrt{w} + \frac{1}{\sqrt{w}}\right)\bar{S}^1\sigma_1 + \frac{i}{2}\left(\sqrt{w} - \frac{1}{\sqrt{w}}\right)\bar{S}^2\sigma_2 \tag{52}
$$

One can notice that (48) and (52) are essentially the same. In particular, Lax operator (52) satisfies the quadratic Poisson relation (10) with the same trigonometric r-matrix (49). Indeed, these two Lax matrix are related by the simple transformation

$$
L_{ds} = -\sin\left(\pi\xi\sigma_2\right) L_{XXZ} \tag{53}
$$

w being identified with $e^{2i\xi}$ and $\bar{S}^0$, $\bar{S}^1$, $\bar{S}^2$, $\bar{S}^3$ with $\hat{S}^2$, $\hat{S}^3$, $\hat{S}^0$, $\hat{S}^1$ respectively. Let us also note that Lax operator (52) is nothing but the L-operator of the lattice Sine-Gordon model.

The determinant $\det_{2\times 2}\hat{L}(\xi)$ is equal to

$$
\begin{aligned}
\det_{2\times 2}\hat{L}(\xi) &= \hat{S}_0^2 + \sum_{a=1}^{3} e_a\hat{S}_a^2 - \wp(\xi)\sum_{a=1}^{3}\hat{S}_a^2 = \\
&= K - M^2\wp(\xi) = K - M^2 x
\end{aligned}
\tag{54}
$$

where

$$
K = \hat{S}_0^2 + \sum_{a=1}^{3} e_a(\tau)\hat{S}_a^2 \qquad M^2 = \sum_{a=1}^{3}\hat{S}_a^2 \tag{55}
$$

are the Casimir operators of the Sklyanin algebra (i.e. Poisson commuting with all the generators $\hat{S}^0$, $\hat{S}^1$, $\hat{S}^2$, $\hat{S}^3$). The determinant of the monodromy matrix (11) is

$$
Q(\xi) = \det_{2\times 2} T_{N_c}(\xi) = \prod_{i=1}^{N_c}\det_{2\times 2}\hat{L}(\xi-\xi_i) = \prod_{i=1}^{N_c}\left(K_i - M_i^2\wp(\xi-\xi_i)\right) \tag{56}
$$

while the trace $\mathcal{P}(\xi) = \frac{1}{2}\mathrm{Tr}T_{N_c}(\xi)$ generates mutually Poisson-commuting Hamiltonians, since

$$
\{\mathrm{Tr}T_{N_c}(\xi),\ \mathrm{Tr}T_{N_c}(\xi')\} = 0 \tag{57}
$$

For example, in the case of the *homogeneous* chain (all $\xi_i = 0$ in (56)) $\mathrm{Tr}T_{N_c}(\xi)$ is a combination of the polynomials:

$$
\mathcal{P}(\xi) = Pol^{(1)}_{\left[\frac{N_c}{2}\right]}(x) + yPol^{(2)}_{\left[\frac{N_c-3}{2}\right]}(x), \tag{58}
$$

where $\left[\frac{N_c}{2}\right]$ is integral part of $\frac{N_c}{2}$, and the coefficients of $Pol^{(1)}$ and $Pol^{(2)}$ are Hamiltonians of the XYZ model [1]. As a result, the spectral equation (38) for the XYZ model acquires the form:

$$\tilde{w} + \frac{Q(\xi)}{\tilde{w}} = 2\mathcal{P}(\xi), \tag{59}$$

where for the *homogeneous* chain $\mathcal{P}$ and Q are polynomials in $x = \wp(\xi)$ and $y = \frac{1}{2}\wp'(\xi)$. Eq. (59) describes the double covering of the elliptic curve $E(\tau)$: with generic point $\xi \in E(\tau)$ one associates the two points of $\mathcal{C}^{XYZ}$, labeled by two roots $w_\pm$ of equation (59). The ramification points correspond to $\tilde{w}_+ = \tilde{w}_- = \pm\sqrt{Q}$, or $Y = \frac{1}{2}\left(\tilde{w} - \frac{Q}{\tilde{w}}\right) = \sqrt{\mathcal{P}^2 - Q} = 0$.

The curve (59) is in fact similar to that of $N_c = 2$ Calogero-Moser system (31). The difference is that now $x = \infty$ is *not* a branch point, therefore, the number of cuts on the both copies of $E(\tau)$ is N_c and the genus of the spectral curve is $N_c + 1$.

Rewriting analytically $\mathcal{C}^{XYZ}$ as a system of equations

$$\begin{aligned} y^2 &= \prod_{a=1}^{3}(x - e_a), \\ Y^2 &= \mathcal{P}^2 - Q \end{aligned} \tag{60}$$

the set of holomorphic 1-differentials on $\mathcal{C}^{XYZ}$ can be chosen as

$$\begin{aligned} v &= \frac{dx}{y}, \\ V_\alpha &= \frac{x^\alpha dx}{yY} \quad \alpha = 0, \ldots, \left[\frac{N_c}{2}\right], \\ \tilde{V}_\beta &= \frac{x^\beta dx}{Y} \quad \beta = 0, \ldots, \left[\frac{N_c - 3}{2}\right] \end{aligned} \tag{61}$$

with the total number of holomorphic 1-differentials $1 + \left(\left[\frac{N_c}{2}\right] + 1\right) + \left(\left[\frac{N_c-3}{2}\right] + 1\right) = N_c + 1$ being equal to the genus of $\mathcal{C}^{XYZ}$.

4 Generating 1-form

Given the spectral curve and the integrable system one can immediately write down the "generating" 1-differential dS which obeys the basic defining property (1). For the Toda chain it can be chosen in two different ways

$$\begin{aligned} d\Sigma^{TC} \cong d\lambda \log w \qquad dS^{TC} \cong \lambda\frac{dw}{w} \\ d\Sigma^{TC} = -dS^{TC} + df^{TC} \end{aligned} \tag{62}$$

Both $d\Sigma^{TC}$ and dS^{TC} obey the basic property (1) and, while f^{TC} itself is *not* its variation, $\delta f^{TC} = \lambda\frac{\delta w}{w}$ appears to be a (meromorphic) single-valued function on $\mathcal{C}^{TC}$.

[1] For the *inhomogeneous* chain the explicit expression for the trace is more sophisticated: one should make use of the formulas like

$$\wp(\xi - \xi_i) = \left(\frac{\wp'(\xi) + \wp'(\xi_i)}{\wp(\xi) - \wp(\xi_i)}\right)^2 - \wp(\xi) - \wp(\xi_i) = 4\left(\frac{y + y_i}{x - x_i}\right)^2 - x - x_i$$

In the XXX case [1], one has almost the same formulas as (62)

$$d\Sigma^{XXX} \cong d\lambda \log w \qquad dS^{XXX} \cong \lambda \frac{dw}{w}$$
$$d\Sigma^{XXX} = -dS^{XXX} + df^{XXX} \tag{63}$$
$$w = \frac{\tilde{w}}{\sqrt{\det T_{N_c}(\lambda)}}$$

For the XYZ model (59) the generating 1-form(s) dS^{XYZ} can be defined as

$$d\Sigma^{XYZ} \cong d\xi \cdot \log w$$
$$dS^{XYZ} \cong \xi \frac{dw}{w} = -d\Sigma^{XYZ} + d(\xi \log w) \tag{64}$$

Now, under the variation of moduli (which are all contained in $\mathcal{P}$, while Q is moduli independent),

$$\delta(d\Sigma^{XYZ}) \cong \frac{\delta w}{w} d\xi = \frac{\delta \mathcal{P}(\xi)}{\sqrt{\mathcal{P}(\xi)^2 - Q(\xi)}} d\xi = \frac{dx}{yY} \delta \mathcal{P} \tag{65}$$

and, according to (6), the r.h.s. is a *holomorphic* 1-differential on the spectral curve (59).

The singularities of $d\Sigma^{XYZ}$ are located at the points where $w = 0$ or $w = \infty$, i.e. at zeroes of $Q(\xi)$ or poles of $\mathcal{P}(\xi)$. In the vicinity of a singular point, $d\Sigma^{XYZ}$ is not single-valued but acquires addition $2\pi i d\xi$ when circling around this point. The difference between $d\Sigma$ and dS is again a total derivative, but $\delta f^{XYZ} = \xi \frac{\delta w}{w}$ is not a single-valued function. In contrast to $d\Sigma^{XYZ}$, dS^{XYZ} has simple poles at $w = 0, \infty$ with the residues $\xi|_{w=0,\infty}$, which are defined modulo $1, \tau$. Moreover, dS^{XYZ} itself is multivalued: it changes by $(1, \tau) \times \frac{dw}{w}$ when circling along non-contractable cycles on $E(\tau)$.

Naively, neither $d\Sigma^{XYZ}$ nor dS^{XYZ} can play the role of the Seiberg-Witten 1-form – which is believed to possess well-defined residues, interpreted as masses of the matter hypermultiplets [9].

In the simplest example $N_c = 2$, the second equation in (60) is

$$Y^2 = \mathcal{P}^2 - Q = (H_0 - H_2 x)^2 - (K_1 - M_1^2 x)(K_2 - M_2^2 x)$$
$$\equiv A(x - x_1)(x - x_2) \tag{66}$$

It is a curve of genus $N_c + 1 = 3$, obtained by gluing two copies of $E(\tau)$ along two cuts: between $x = x_1$ and $x = x_2$ on every of two sheets of $E(\tau)$. In (66)

$$H_0 = \hat{S}_1^0 \hat{S}_2^0 + \sum_{a=1}^{3} e_a \hat{S}_1^a \hat{S}_2^a \qquad H_2 = \sum_{a=1}^{3} \hat{S}_1^a \hat{S}_2^a \tag{67}$$

and, comparing with (55), it is natural to represent

$$H_2 = M_1 M_2 \cosh h \tag{68}$$

Such a separation of the Casimir (M) and moduli (h) dependence is implied by considering the various limits: conformal one – with all $M_i \to 0$ and a "dynamical transmutation" regime when some $M_i \to \infty$ along with $\tau \to +i\infty$.

When $\tau \to +i\infty$ or $q = e^{i\pi\tau} \to 0$, the ramification points e_1 and e_2 collide: $e_1 - e_2 = 16q + \mathcal{O}(q^3)$ and the proper coordinates on $\mathcal{C}^{XYZ}$ are $x = -\frac{1}{3} + q\check{x}$, $y = q\check{y}$. Then, equation (23) for $E(\tau)$ turns into:

$$\check{y}^2 = \check{x}^2 - 1 \tag{69}$$

describing the double-sheet covering of CP^1, which is again CP^1. The canonical holomorphic 1-form $d\xi = 2\frac{dx}{y}$ turns into

$$2\frac{d\check{x}}{\check{y}} = 2\frac{d\check{x}}{\sqrt{\check{x}^2-1}} = 2\frac{dz}{z} \tag{70}$$

where $\check{x} = z + z^{-1}$.

The *double*-scaling limit assumes that the ramification points x_1 and x_2 also behave in a special way as $q \to 0$. Namely, let

$$x_i = -\frac{1}{3} + q\check{x}_i \tag{71}$$

Now, rescaling $Y = q\check{Y}$, one gets for $\mathcal{C}^{XYZ}$ in the double-scaling limit

$$\begin{gathered} \check{y}^2 = \check{x}^2 - 1, \\ \check{Y}^2 = A(\check{x} - \check{x}_1)(\check{x} - \check{x}_2) \end{gathered} \tag{72}$$

These equations describe two copies of CP^1 glued together along the two cuts (between $\check{x} = \check{A}_1$ and $\check{x} = \check{A}_2$ on every of two sheets) – i.e. this is an elliptic curve (torus) of genus 1.

The generating 1-form

$$d\Sigma^{XYZ} \cong d\xi \cdot \log w \to d\Sigma^{TC} \cong \frac{dz}{z}\log w \tag{73}$$

For generic N_c the multi-scaling limit can be performed in a similar to (71) way implying that the full spectral curve of genus $N_c + 1$ – a double covering of $E(\tau)$ – degenerates into the double-covering of CP^1, which is of genus $N_c - 1$ and is associated with the Toda-chain system. The generating differentials $d\Sigma^{XYZ}$ and dS^{XYZ} also turn into the corresponding Toda-chain generating 1-forms (62).

5 Comments

We discussed some elementary results about the XYZ chain from the perspective of the Seiberg-Witten exact solutions. In this framework, one associates with every (finite-gap) integrable system a family of spectral curves (so that the integrals of motion play the role of moduli of the complex structures) and (the cohomology classes of) the generating 1-form dS, satisfying (1), which can be used to construct the "periods" a, a_D and the prepotential. According to [2], the prepotential for the $4d$ $\mathcal{N} = 2$ SQCD coincides with (the logarithm of) the τ-function of the associated integrable model.

One can naturally assume that the XYZ chain, which is an elliptic deformation of the XXX chain known to describe $\mathcal{N} = 2$ supersymmetric QCD with $N_f < 2N_c$ [1], can be associated with the $N_f = 2N_c$ case. This would provide a description of the conformal (UV-finite) supersymmetric QCD, differing from the conventional one [9, 10]. However, as demonstrated in the present text, there are several serious differences between XYZ and XXX models, which should be kept in mind. Let us list some of them.

1) Normally, there are two natural ways to introduce $d^{-1}(symplectic\ form)$: as pdq and qdp – these are represented by the meromorphic 1-differentials dS and $d\Sigma$ in the main text. Usually, *both* satisfy (1); $d\Sigma$ has no simple poles, but is not single-valued on $\mathcal{C}$, while dS is single-valued and possesses simple poles (of course, in general $dS \not\cong d\Sigma$). The proper generating 1-form is dS. However, in the XYZ case, *both* dS and $d\Sigma$ are

not single-valued. Moreover, the residues of dS – identified with masses of the matter hypermultiplets in the framework of [9] – are defined only modulo $(1,\tau)$.

2) As $\tau \to i\infty$, the XYZ model turns into XXZ rather than XXX one. This makes the description of the "dimensional transmutation" regime rather tricky.

3) Starting from the spectral curve (5) for the Toda-chain (pure $\mathcal{N}=2$ SYM), the Calogero-Moser deformation is associated with "elliptization" of the w-variable, while the XYZ-deformation – with that of the λ-variable. It is again nontrivial to reformulate the theory in such a way that the both deformations become of the same nature. One of the most naive pictures would associate the Hitchin-type (Calogero) models with the "insertion" of an $SL(N_c)$-orbit at one puncture on the elliptic curve, while the spin-chain (XYZ) models – with the $SL(2)$-orbits at N_c punctures. Alternatively, one can say that the XYZ-type "elliptization", while looking local (i.e. L-operators at every site are deformed independently), is in fact a global one (all the L's can be elliptized only simultaneously, with the same r-matrix and τ), – but this is not clearly reflected in existing *formalism*, discussed in this paper. Moreover, the proper formalism should naturally allow one to include any simple Lie group (not only $SL(N_c)$) and any representations (not only adjoint or fundamental).

Already these comments are enough to demonstrate that the hottest issues of integrability and quantum-group theory (like notions of elliptic groups and dynamical R-matrices) can be of immediate importance for the Seiberg-Witten (and generic duality) theory. These subjects, however, remain beyond the scope of this note.

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